Worksheet 7

Contents

To accompany Chapter 4.2 Fourier transforms of commonly occurring signals

This worksheet can be downloaded as a PDF file. We will step through this worksheet in class.

An annotatable copy of the notes for this presentation will be distributed before the second class meeting as **Worksheet 7** in the **Week 5: Classroom Activities** section of the Canvas site. I will also distribute a copy to your personal **Worksheets** section of the **OneNote Class Notebook** so that you can add your own notes using OneNote.

You are expected to have at least watched the video presentation of 4.2: Fourier transforms of commonly occurring signals of the notes before coming to class. If you haven't watch it afterwards!

After class, the lecture recording and the annotated version of the worksheets will be made available through Canvas.

Reminder of the Definitions

Last time we derived the Fourier Transform by evaluating what would happen when a

Skip to main content

The Fourier Transform

Used to convert a function of time $f(t)$ to a function of radian frequency $F(\omega)$:

$$\mathcal{F}\left\{f(t)\right\} = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\, dt = F(\omega).$$

The Inverse Fourier Transform

Used to convert a function of frequency $F(\omega)$ to a function of time $f(t)$:

$$\mathcal{F}^{-1}\left\{F(\omega)\right\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\, d\omega = f(t).$$

Note, the factor 2π is introduced because we are changing units from radians/second to seconds.

Duality of the transform

Note the similarity of the Fourier and its Inverse.

This has important consequences in filter design and later when we consider sampled data systems.

Table of Common Fourier Transform Pairs

This table is adapted from Table 8.9 of Karris. See also: Wikibooks: Engineering Tables/Fourier Transform Table and Fourier Transform—WolframMathworld for more complete references.

Skip to main content

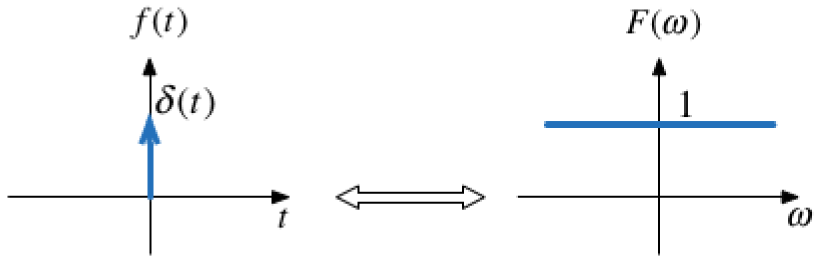
	Name	$f(t)$	$F(\omega)$	Remarks
1.	Dirac delta	$\delta(t)$	1	Constant energy at *all* frequencies.
2.	Time sample	$\delta(t-t_0)$	$e^{-j\omega t_0}$	
3.	Phase shift	$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$	
4.	*Signum*	$\operatorname{sgn} t$	$\frac{2}{j\omega}$	also known as sign function
5.	Unit step	$u_0(t)$	$\frac{1}{j\omega}+\pi\delta(\omega)$	
6.	Cosine	$\cos\omega_0 t$	$\pi\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right]$	
7.	Sine	$\sin\omega_0 t$	$-j\pi\left[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)\right]$	
8.	Single pole	$e^{-at}u_0(t)$	$\frac{1}{j\omega+a}$	$a>0$
9.	Double pole	$te^{-at}u_0(t)$	$\frac{1}{(j\omega+a)^2}$	$a>0$
10.	Complex pole (cosine component)	$e^{-at}\cos\omega_0 t\ u_0(t)$	$\frac{j\omega+a}{(j\omega+a)^2+\omega_0^2}$	$a>0$
11.	Complex pole (sine component)	$e^{-at}\sin\omega_0 t\ u_0(t)$	$\frac{\omega_0}{(j\omega+a)^2+\omega_0^2}$	$a>0$

Some Selected Fourier Transforms

The Dirac Delta

$$\delta(t) \Leftrightarrow 1$$

Skip to main content



Proof: uses sampling and sifting properties of $\delta(t)$.

MATLAB:

```
syms t omega omega_0 t0;
u0(t) = heaviside(t); % useful utility function
fourier(dirac(t))
```

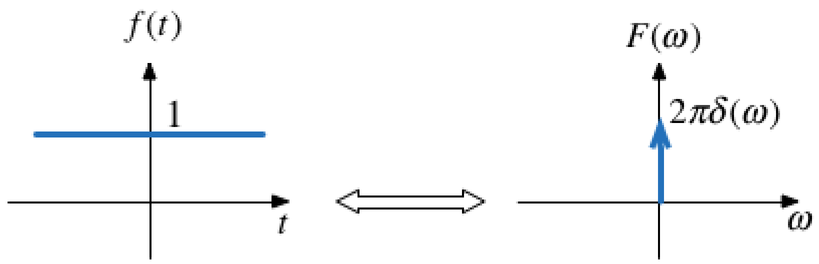
Related:

$$\delta(t - t_0) \Leftrightarrow e^{-j\omega t_0}$$

```
fourier(dirac(t - t0),omega)
```

DC

$$1 \Leftrightarrow 2\pi\delta(\omega)$$



MATLAB:

```
A = sym(1); % take one to be a symbol
fourier(A,omega)
```

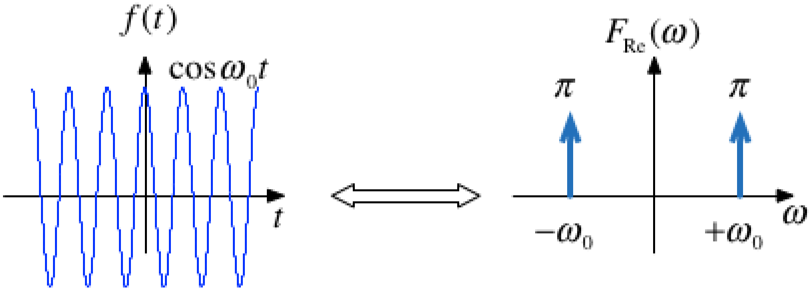
Related by frequency shifting property:

Skip to main content

$$e^{j\omega_0 t} \Leftrightarrow 2\pi\delta(\omega-\omega_0)$$

Cosine (Sinewave with even symmetry)

$$\cos(t) = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right) \Leftrightarrow \pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)$$



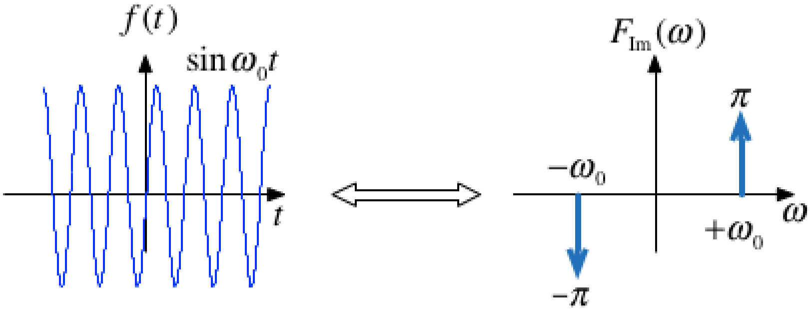
Note: $f(t)$ is real and even. $F(\omega)$ is also real and even.

MATLAB:

```
fourier(cos(omega_0*t),omega)
```

Sinewave

$$\sin(t) = \frac{1}{j2}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right) \Leftrightarrow -j\pi\delta(\omega-\omega_0) + j\pi\delta(\omega+\omega_0)$$



Note: $f(t)$ is real and odd. $F(\omega)$ is imaginary and odd.

MATLAB:

Skip to main content

```
fourier(sin(omega_0*t),omega)
```

Signum (Sign)

The signum function is a function whose value is equal to

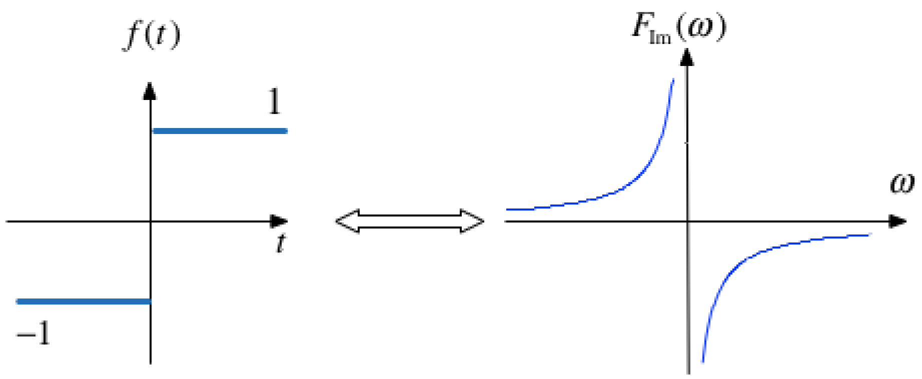
$$\operatorname{sgn} t = \begin{cases} -1 \; t < 0 \\ 0 \; x = 0 \\ +1 \; t > 0 \end{cases}$$

MATLAB:

```
fourier(sign(t),omega)
```

The transform is:

$$\operatorname{sgn} t = u_0(t) - u_0(-t) = \frac{2}{j\omega}$$



This function is often used to model a *voltage comparitor* in circuits.

Example 4: Unit Step

Use the signum function to show that

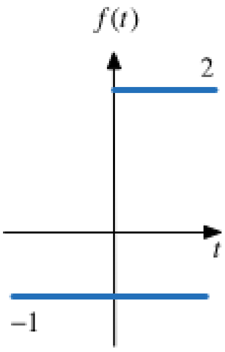
$$\mathcal{F}\left\{u_0(t)\right\} = \pi\delta(\omega) + \frac{1}{j\omega}$$

Skip to main content

Clue

Define

$$\operatorname{sgn} t = 2u_0(t) - 1$$



Does that help?

MATLAB:

```
fourier(u0(t),omega)
```

Example 5

Use the results derived so far to show that

$$e^{j\omega_0 t}u_0(t) \Leftrightarrow \pi\delta(\omega-\omega_0) + \frac{1}{j(\omega-\omega_0)}$$

Hint: linearity plus frequency shift property.

Example 6

Use the results derived so far to show that

$$\sin\omega_0 t\ u_0(t) \Leftrightarrow \frac{\pi}{j2}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]+\frac{\omega_0}{\omega_0^2-\omega^2}$$

Hint: Euler's formula plus solution to example 5.

Important note: the equivalent example in Karris (Section 8.4.9 Eq. 8.75 pp 8-23—8-24) is wrong!

See worked solution in OneNote for corrected proof.

Skip to main content

Example 7

Use the result of Example 3 to determine the Fourier transform of $\cos\omega_0 t\ u_0(t)$.

Answer

$$\cos\omega_0 t\ u_0(t) \Leftrightarrow \frac{\pi}{2}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)] + \frac{j\omega}{\omega_0^2-\omega^2}$$

Derivation of the Fourier Transform from the Laplace Transform

If a signal is a function of time $f(t)$ which is zero for $t \leq 0$, we can obtain the Fourier transform from the Laplace transform by substituting s by $j\omega$.

Example 8: Single Pole Filter

Given that

$$\mathcal{L}\left\{e^{-at}u_0(t)\right\} = \frac{1}{s+a}$$

Compute

Skip to main content

$$\mathcal{F}\left\{e^{-at}u_0(t)\right\}$$

Example 9: Complex Pole Pair cos term

Given that

$$\mathcal{L}\left\{e^{-at}\cos\omega_0 t\ u_0(t)\right\} = \frac{s+a}{(s+a)^2+\omega_0^2}$$

Compute

$$\mathcal{F}\left\{e^{-at}\cos\omega_0 t\ u_0(t)\right\}$$

Skip to main content

Fourier Transforms of Common Signals

We shall conclude this session by computing as many of the the Fourier transform of some common signals as we have time for.

- rectangular pulse
- triangular pulse
- periodic time function
- unit impulse train (model of regular sampling)

Previous
Worksheet 6

Next
Worksheet 8